Lecture 15.

<u>Recall.</u> M open subset in $\mathbb{R}^m$, $X_1, \dots, X_r$ real vector fields in M, $\mathfrak{g}$ the Lie algebra generated by them.

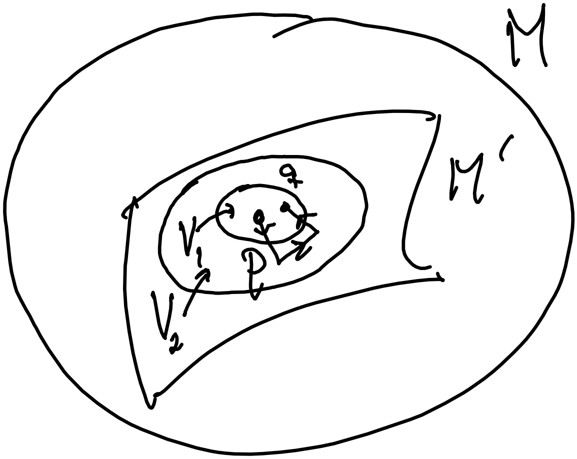
<u>Local Nagano.</u> Assume $X_1, \dots, X_r$ are $\underline{C^\omega}$. For every $p \in M$, $\exists!\, M' \subseteq U \overset{\text{open}}{\subseteq} M$, s.t. $p \in M'$ and $T_q M' = \mathfrak{g}(q)$, $\forall q \in M'$ (M' = Local Nagano Leaf).

Not true in general if $X_1, \dots, X_r$ are only C^∞. Ex. Consider $X_1 = \frac{\partial}{\partial x}$, $X_2 = e^{-\frac{1}{x^2}} \frac{\partial}{\partial y}$ in $\mathbb{R}^2$.

<u>Sussman's Thm.</u> $M, X_1, \dots, X_r, \mathfrak{g}$ as in LNT, but only $\underline{C^\infty}$ (not C^ω). Then, for every $p \in M$, $\exists$ open nbhd $U \subseteq M$, a "unique" $M' \subseteq U$ w/ $p \in M'$ s.t.

$$\mathfrak{g}(q) \subseteq T_q M', \ \forall q \in M'$$

Moreover, $\forall q \in V_1 \subseteq V_2 \subseteq M'$ there is a polygonal path P of curves in V_2 s.t. the tangents of the curves are spanned by $X_1, \dots, X_r$ and P connects p to q.



- M' in Sussman's Thm is called the <u>Sussman orbit</u>. If $X_1, \dots, X_r$ are C^ω, then the Sussman orbit coincides with the local Nagano leaf.

Real hypersurfaces / CR mflds of hypersurface type

We now specialize to the hypersurface ($d =$ CR codim $M = 1$) case. Thus, $(M, \mathcal{V})$ is a CR mfld of CR dim $M = n$, CR codim $M = 1 \Rightarrow$ $\dim_{\mathbb{R}} M = 2n+1$. As before, $H = (\mathcal{V} \oplus \overline{\mathcal{V}}) \cap TM$:

Let $\theta \neq 0$ be a real 1-form s.t., for each $p \in M$,

$$\theta^{\perp} = \{ X_p \in T_p M : \theta(X) = 0 \} = H, \text{ or equiv.,}$$

$$\theta(Z) = \theta(\overline{Z}) = 0, \quad \forall Z_p \in \mathcal{V}_p.$$

Let $Z_1, \ldots, Z_n$ be a local frame for $\mathcal{V}$ on some open set in M. We take a real vector field T s.t. $\theta(T) = 1 \Rightarrow$ $\{Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n, T\}$ is a local frame for $\mathbb{C}TM$.

Levi form.

<u>Def.</u> The <u>Levi form</u> of M at $p \in M$ is the Hermitian form $\mathcal{V} \times \mathcal{V} \to \mathbb{C}$:

$$\mathcal{L}^{\theta}_p(Z_p, W_p) := \frac{1}{2i}\,\theta([Z, \bar{W}])\big|_p \quad Z_p, W_p \in \mathcal{V}_p,$$

where Z, W are local sections of $\mathcal{V}$ extending Z_p, W_p, respectively.

Def. seems to depend on the extensions Z, W and the choice of θ. Note any other choice of θ is related by $\tilde{\theta} = a\theta$, where $a \neq 0$ is real-valued function.

<u>Prop.</u> The Levi form $\mathcal{L}^{\theta}_p$ does not depend on the extensions Z, W. If $\tilde{\theta} = a\theta$, then

$$\mathcal{L}^{\tilde{\theta}}_p = a(p)\, \mathcal{L}^{\theta}_p.$$

<u>Pf.</u> The last identity $\mathcal{L}^{\tilde{\theta}}_p = a(p)\, \mathcal{L}^{\theta}_p$ is obvious from the def. once we prove that $\mathcal{L}^{\theta}_p$ is well defined (i.e. first part).